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Julia sets associated with the Potts model on the Bethe lattice and other recursively solved systems

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Abstract

The q-state Potts model on the Bethe lattice has been investigated by a number of individuals over the last several decades with some of the preceding studies taking a dynamical systems perspective. However, other than for the special case of the $q = 2$ Potts model, i.e. the Ising model, the Julia sets of the discrete dynamical systems associated with the Potts model on the Bethe lattice have been largely ignored. We look at these sets and find an interesting connection between the phase transition temperature and the dimension of the Julia set. In particular we find the dimension of the Julia set to be a minimum at a phase transition. Furthermore we show this property is not restricted to Potts models on Bethe lattices. This adds to a number of other special properties present at temperatures where a phase transition occurs.

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1. Introduction

The q-state Potts model of which the Ising model is a special case has been of great interest for some time. Exact solutions on the standard lattices such as the square or honeycomb lattices are in general extremely rare and due to this several groups of investigators have studied the properties of this class of system on the Bethe lattice as an approximation to the behaviour on the standard lattices as well as being of interest in their own right. This continues to the present time as evidenced by the very recent article, in this journal, of Wagner *et al* [1]. Many of these investigators have used a dynamical systems approach. This was first done for the special case of the Ising model (see Eggarter [2] and Thompson [3]) but was extended to the full set of Potts models in a series of papers by de Aguiar *et al* [4] and references therein as well as by Ananikyan *et al* [5] and references therein. Other approaches have been presented, in particular Peruggi *et al* [6] have used an approach relying on the construction of a translationally invariant probability measure and from it an expression for the free energy which allowed them to determine where phase transitions occurred.

In the following we use a dynamical systems approach to study the Potts model on the Bethe lattice and focus attention on the Julia set associated with the map describing the system. In particular we look at the connection between the behaviour of the Julia set and the behaviour of the system in terms of its phase diagram. In doing so we find what we believe to be a very interesting and simple situation: the fractal dimension of the Julia set, more accurately the box counting dimension, is a minimum at exactly the temperature at which the phase transition takes place when the free energy approach is used. The Julia set associated with the mapping for the special case of the Ising model has been studied in some detail, see [7] and [8] but even here only for the case when the external magnetic field is zero in which case the Julia set is the unit circle or some multi-fractal set lying on the unit circle. We want to vary the external magnetic field at temperatures below the critical temperature and see the change in the characteristics of the Julia set.

The extent to which such a simple property is true is of interest. To shed some light on this question we look at a second system, an Ising spin system where not only nearest neighbour, pair interactions are present, but interactions involving three sites are also present. The system has a rather complex phase diagram because of the presence of the interaction involving more than two sites. The phase transitions occur for values of the magnetic field not equal to zero. The system was initially chosen because the phase diagram is known exactly. In this case we find the same connection between the minimum fractal dimension of the Julia set associated with this map and the occurrence of phase transitions for this system.

This connection between the dimension of the Julia set and a phase transition is augmented by another very special connection. This connection is between the stability of the fixed points of the map and the occurrence of a phase transition as presented in [9]. Explicitly the temperature at which a phase transition occurs based on free energy considerations coincides exactly with the temperature at which one fixed point becomes more stable than another as the temperature is varied. By the stability of a fixed point we mean the absolute value of the derivative of the governing map evaluated at the fixed point value. The closer to zero the more stable the fixed point.

In the following section we present our Julia set results for the Potts model on the Bethe lattice and this is followed in section 3 by results for the multi-site interaction Ising model mentioned earlier. Section 4 contains some concluding remarks.

2. Potts model on the Bethe lattice

The Hamiltonian for the general q-state Potts model is

$$H = -\sum_{(i,j)} J\delta(\sigma_i, \sigma_j) - \sum_i h\delta(\sigma_i, 1) \tag{1}$$

where σ_i is the spin variable on the ith site and can take on the values $1, 2, \ldots, q$, $\delta(x, y)$ is the usual Kronecker delta, h is the external magnetic field, the first sum is over all nearest neighbour sites, and the second sum is over all sites of the lattice. We consider the system on a Bethe lattice or Cayley tree of branching ratio k. Such a tree can be thought of as being constructed in steps by connecting branches as illustrated in figure 1 where we have first, second and third generation branches with $k = 2$. As stated above this system has been studied by a number of authors from the point of view of a recursive system which has associated with it a map, in

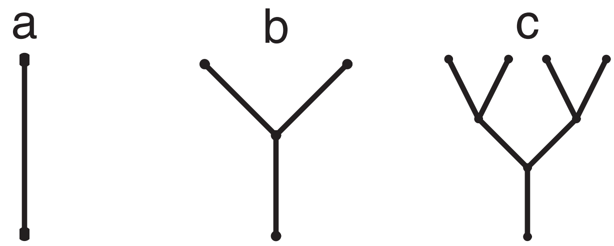

Figure 1. First, second and third generation branches for a Cayley tree with $k = 2$.

this case what we denote as the 'Potts–Bethe' map. The map is the following:

$$z_{n+1} = \frac{abz_n^k + (q-1)}{az_n^k + b + (q-2)} \tag{2}$$

where z_n is a ratio of the partition function of an nth generation branch with the spin on the base site of the branch equal to one to the partition function with the spin not equal to one and where $a = \exp[\beta h]$, $b = \exp[\beta J]$ and $\beta = 1/T$, where T is the temperature. Many authors have derived this map, see e.g. equation (9) [5] or (4) [1]. The behaviour of the system is governed by the behaviour of the variable z in (2) through the equation

$$\langle \delta(\sigma_0, 1)\rangle_n = \frac{az_n^{k+1}}{az_n^{k+1} + (q-1)} \tag{3}$$

with the brackets denoting the usual thermal average and the spin variable σ_0 is the spin variable on the central site of the lattice. The central site is a site where $k + 1$ nth generation branches have been attached to that site. One is interested in letting $n \to \infty$, i.e. in the thermodynamic limit. When $n \to \infty$ in (2) one reaches a fixed point if any attracting fixed points exist. The fixed point behaviour is therefore critical in determining the behaviour of the system.

We will consider only the ferromagnetic case, i.e. where $J > 0$. For simplicity we start by taking $h = 0$, $q > 1$, and $q \neq 2$ (we will mention the special Ising case later). The behaviour when $q \leqslant 1$ has a number of special properties, see [10] and [11]. With these restrictions one has at high temperatures a single, positive, real valued fixed point (due to the definition of z only positive, real valued fixed points have any physical meaning), $z = 1$, that is attracting or in other words stable. As the temperature is lowered a tangent bifurcation occurs at temperature T_1. As characteristic of a tangent bifurcation two new real valued fixed points (both it turns out are positive) are created which we denote as z_+ and z_-. One, z_+, is an attracting fixed point and the other, z_-, is repelling. Lowering the temperature still further a transcritical bifucation occurs at a temperature T_2 where the fixed $z = 1$ goes from being attracting to repelling and the exact opposite occurs with z_-.

As an aside we mention that while the above has been found to be the situation for a number of specific cases where the values of q and k have been set it has not been proven that the possibility of more than three positive, real valued fixed points cannot occur. Finding the fixed points of (2) results in a $k + 1$ degree polynomial guaranteeing $k + 1$ fixed points, some complex and some real. Nevertheless a straightforward application of Descartes's rule of signs [12] guarantees there will be a maximum of three positive, real valued fixed points. Furthermore if there are not three the rule requires there be only one.

If one takes a completely dynamical systems approach to determining the behaviour of the Potts model system being studied one iterates (2) starting with $n = 0$, the value z_0 being the boundary condition. For $z_0 > 1$ one obtains a first-order phase transition somewhere between

T_1 and T_2, the exact value is dependent on the exact value of z_0. If $0 < z_0 < 1$ (remember a negative z has no physical meaning) then one obtains a second-order or continuous phase transition at T_2, see [13]. If one takes the standard statistical mechanics approach one should calculate a free energy and the system goes to the state corresponding to the minimum free energy. This was the approach of Peruggi *et al* [6]. There it was found that there is a first-order phase transition at

$$b_c = \frac{q-2}{(q-1)^{(k-1)/(k+1)} - 1} \tag{4}$$

that corresponds to a temperature between T_1 and T_2 which we denote as T_c. It is at this temperature that the two very special and very simple properties, mentioned in the introduction, occur. One involves the stability of the fixed point the other a property of the Julia set. We have previously shown that the stability of each attracting fixed point at the temperature T_c are equal and that at a temperature $T > T_c$ ($T < T_c$) the fixed point $z = 1$ (z_+) is the more stable fixed point. Hence in addition to the general statistical mechanics statement, that as one decreases the value of the temperature the system jumps from one phase with the minimum free energy to another with the new minimum free energy, we have the parallel dynamical systems statement, that as the temperature is decreased the system jumps from a phase corresponding to the most stable fixed point to another phase corresponding to the new most stable fixed point.

In addition to the behaviour regarding the stability of the fixed points at the temperature given by (4) there also is a remarkably simple property of the Julia sets. In particular at T_c the Julia set is a circle and hence has dimension one while at temperatures just above or just below T_c the Julia set has a fractal or box counting dimension greater than one. (Actually the Julia set is multi-fractal but we are only interested in the box counting dimension.) Therefore the phase transition occurs at the minimum of the fractal dimension taken as a function of temperature. One sees this immediately by looking at the Julia sets associated with the Potts–Bethe map. Representative Julia sets of (2) are shown in figure 2 for temperatures above, below and at T_c for the case $q = 9$ and $k = 2$. From equation (4) above we know the phase transition occurs at $b_c = 7$. At this value of b it is very easy to determine the fixed points from equation (2) and they are $z = 1, 2$ and 4. The fixed points $z = 1$, and 4 are stable fixed points and the fixed point $z = 2$ is unstable or repelling. Therefore $z = 2$ is an element of the Julia set and as figure 2(*c*) shows the Julia set is a circle of radius 2 at this value of b. One can show this analytically by taking an arbitrary point in the complex plane of modulus 2, backward or forward iterate the Potts–Bethe map using this value, and show the resulting value will again be of modulus 2 and therefore on the circle of radius 2. The Julia set is invariant to forward and backward images [14].

The box-counting dimensions of the Julia sets of the Potts–Bethe map for the case $q = 9$ and $k = 2$ are given in figure 3 for a number of values of b around $b_c = 7$. The values displayed in the figure are based on estimates using boxes having sides from 1/400 to 1/8000 of a unit interval. Twelve different box sizes were used for each estimate except for the values 6.9, 7 and 7.1 where to gain a little extra accuracy 14 different box sizes were used.

Finally it should be pointed out that while in the above we have varied the value of b about b_c setting $a = 1$ we could just as well have set $b = 7$ and varied the value of a. Doing so produces the Julia sets similar to those shown in figure 2.

The reason we can set b and vary a or set a and vary b for the above system is because the line of phase transitions for the Potts model with $q \neq 2$ is a curved line in the a–b or h–T plane and not a straight line along the $a = 1$ or in other words $h = 0$ axis like it is in the Ising spin case of $q = 2$. In the Ising case for $h = 0$ and some temperature below the critical temperature varying b does not change the Julia set. It is a circle if $h = 0$ and $T < T_c$. Varying h about $h = 0$ produces a series of Julia sets similar to those of figure 2 and so while the $q = 2$

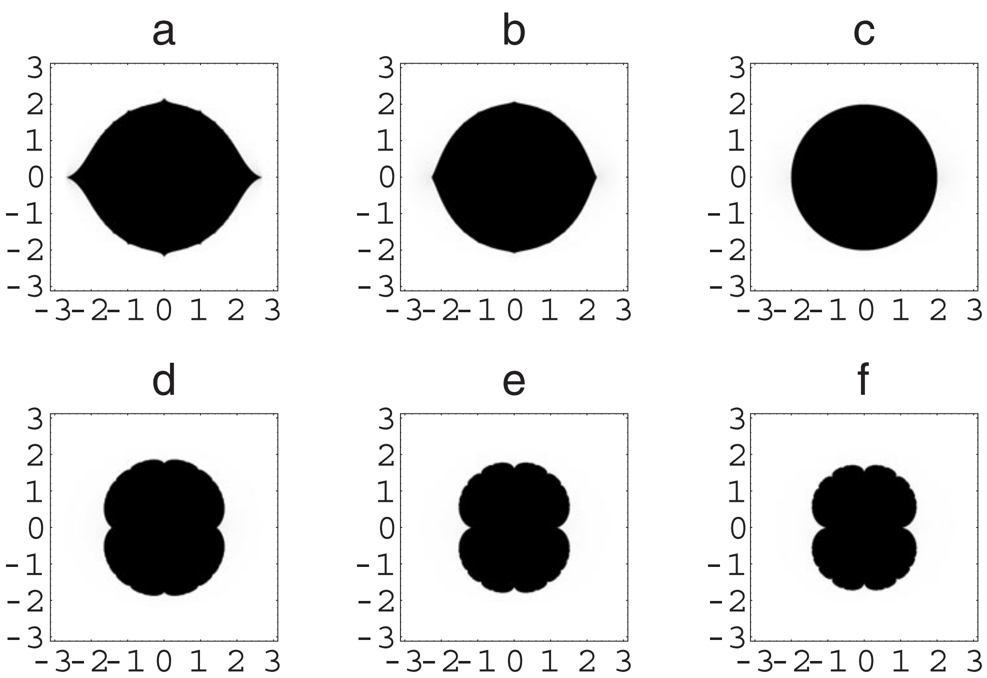

Figure 2. Filled-in Julia sets of equation (2) for $q = 9$, $k = 2$, and $b = 6.67$, 6.8, 7.0, 8.0, 9.0 and 10.0 for (*a*), (*b*), (*c*), (*d*), (*e*) and (*f*) respectively ($b = 7.0$ is the critical value given by equation (4)).

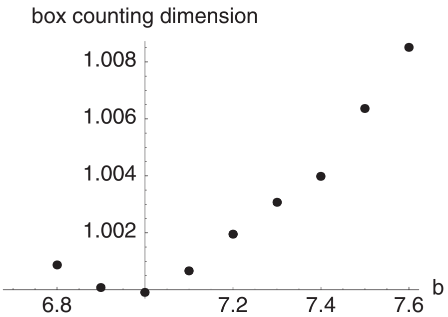

Figure 3. Box-counting dimensions versus b for the Julia sets of equation (2) with $q = 9$ and $k = 2$.

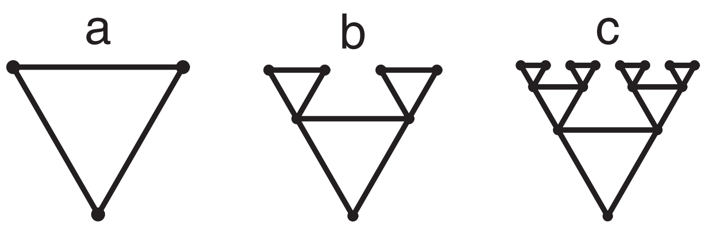

Figure 4. First, second and third generation branches for a Husimi tree used to approximate a triangular lattice system with three site interactions on all down pointing triangles.

case has some differences with the $q \neq 2$ case the basic characteristic of the Julia sets having a minimum box counting dimension at a phase transition point remains to be the case.

3. Multi-site interaction systems

Naturally one would like to know the extent to which this very simple property of the dimension of the Julia set being a minimum at a phase transition is true for systems besides the Potts model on the Bethe lattice. We now present an Ising model system with multi-site interactions and where we have an Husimi tree rather than a Cayley tree, nevertheless the property again holds. An Husimi tree is a tree-like graph built up of polygons rather than edges.

The system is related to an Ising spin model for which Wu and Wu [15] were able to find the exact phase diagram. Their system was on the Kagome lattice rather than any type of tree structure. Being that the exact phase diagram was known and the system has interactions involving more than two sites it was used by the present author as a test case for approximations involving Husimi trees [16].

The original model of Wu and Wu [15] consists of Ising spin variables σ, with $\sigma = \pm 1$, on a Kagome lattice with the Hamiltonian

$$H = -J_3 \sum_{\Delta} \sigma_i \sigma_j \sigma_k - J_2 \sum_{nn} \sigma_i \sigma_j - h \sum_i \sigma_i \tag{5}$$

where the first sum is over all triangles making up the Kagome lattice, the second sum is over all nearest neighbour pairs of sites, and the third sum is over all sites. Phase transitions occur over a surface, for which Wu and Wu found analytical expressions in a three-dimensional space whose axes can be denoted as a, b and c where

$$a = \mathrm{e}^{2\beta h} \qquad b = \mathrm{e}^{2\beta J_2} \qquad c = \mathrm{e}^{2\beta J_3}. \tag{6}$$

An Husimi tree approximation for such a system consists of a Husimi tree made up of triangles with Ising spin variables on the corners. The first three generations of such a tree are shown in figure 4. The map governing the behaviour of such a system is given by

$$z_{n+1} = \frac{a^2 b^2 c z_n^2 + 2a z_n + c}{a^2 z_n^2 + 2ac z_n + b^2}. \tag{7}$$

The map was originally presented in [16] where is was shown that the phase diagram based on a dynamical system approach using boundary conditions gave a good approximation of the exact phase diagram. Later when it was seen that the stability of the fixed point criteria gives better results, results corresponding to those based on free energy analysis, this criteria was applied to this model [9]. It was then found that for any given value of one of the three variables a, b or c defined above the phase diagram of the Husimi tree approximation fell exactly on the phase transition line, given by the exact results, in the plane of the remaining two variables. What the approximation failed to do was to give the exact value for the critical end point, extending the line to points corresponding to higher temperatures than the exact result.

We now examine the behaviour of the Julia sets along one of these lines of phase transition. In particular we will let $a = 1.5$, hence we have a non-zero, positive magnetic field. The phase transition line is then a line in the b–c plane. To restrict ourselves to a particular point on this line we further set the value of b to be 3.0. In then turns out either from the exact result or using the stability of the fixed point criteria that we have a phase transition at $c = 0.519\,743\,165\ldots$. Since this is a point in the three-dimensional space of a, b and c we can vary any one of these variables and examine the subsequent behaviour of the Julia sets about this point. For regions at and around this point we have three real valued, positive fixed points two which are attracting (stable) and the third repelling (unstable). The boundary between the basins of attraction of the two attracting fixed points is the Julia set. This is the same general situation as found in the Potts model case. Julia sets for various values of c with $a = 1.5$ and $b = 3.0$ are shown in

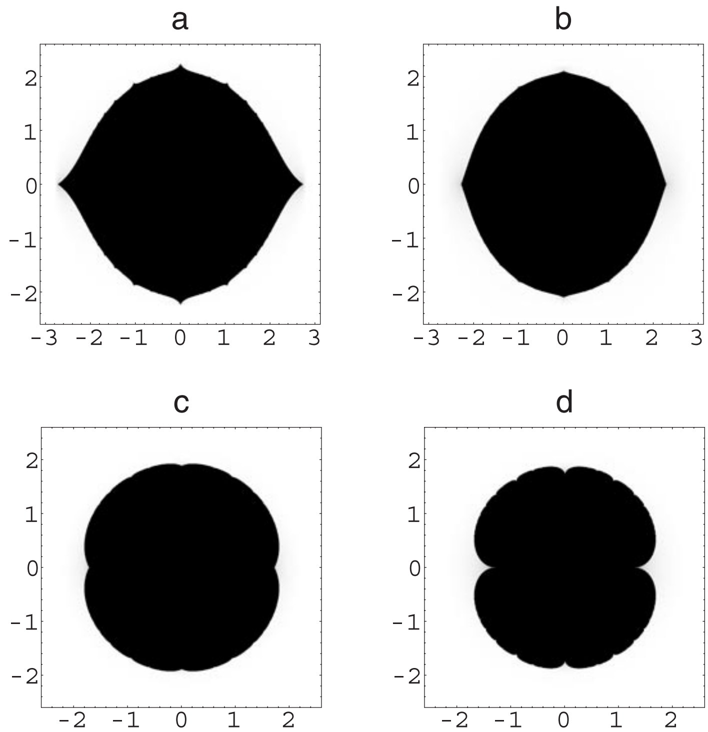

Figure 5. Filled-in Julia sets of equation (7) for $a = 1.5$, $b = 3.0$ and c varying with (*a*) 0.70, (*b*) 0.60, (*c*) 0.45, and (*d*) 0.425.

figure 5. We see that as we move toward the phase transition value the Julia set becomes more circle like and at the phase transition point is simply once again, as in the Potts model case, a circle. On one side of the transition point the Julia set has dimples pointing inward and on the other side dimples pointing outward.

Thus the overall situation once setting two of the three variables involved in the map defined by equation (7) and varying the third looks almost identical to the situation of the previous section involving Potts models even though we have multi-site interactions, three physical parameters rather than two describing the location of a phase transition, and perhaps most importantly a Husimi tree structure rather than Cayley tree structure.

4. Conclusions

Phase transition points have a number of special features when looked at from the standard viewpoint of statistical mechanics. The above shows the same is true from a dynamical systems viewpoint. In particular we have added to the property reported earlier in [9] regarding the stability of the fixed point a very simple property regarding the dimension of Julia sets, that is that the dimension is a minimum at a phase transition point. Unfortunately we have thus far been able to do so only for a smallish class of all possible spin systems and thus showing the property is not specific to a single specific system there is still need for further study. In general the dimension of a Julia set is not a particularly easy quantity to deal with [17] although in our cases the fact at the transition point the Julia set is merely a circle lead to at least immediate visual recognition of the dimension property. Hopefully in the future a direct and analytical connection can be found between the stability of the fixed points, the free energy, and the dimension of Julia sets.

References

[1] Wagner F, Grensing D and Heide J 2000 *J. Phys. A: Math. Gen.* **33** 929
[2] Eggarter T P 1974 *Phys. Rev.* B **9** 2989
[3] Thompson C J 1982 *J. Stat. Phys.* **27** 441
[4] deAguiar F S and Goulart-Rosa S Jr 1992 *Phys. Lett.* A **162** 232
[5] Ananikyan N S and Akheyan A Z 1995 *Zh. Eksp. Teor. Fiz.* **107** 196
Ananikyan N S and Akheyan A Z 1995 *Sov. Phys.–JETP* **80** 105
[6] Peruggi F, diLiberto F and Monroy G 1983 *J. Phys. A: Math. Gen.* **16** 811
[7] Bosco F A and Goulart-Rosa S Jr 1987 *Europhys. Lett.* **4** 1103
[8] Monroe J L 1995 *Europhys. Lett.* **29** 187
[9] Monroe J L 1994 *Phys. Lett.* A **188** 80
[10] Glumac Z and Uzelac K 1994 *J. Phys. A: Math. Gen.* **27** 7709
[11] Monroe J L 1996 *J. Phys. A: Math. Gen.* **29** 5421
[12] Larson R and Hostetler R P 2000 *Precalculus* 5th edn (Boston: Houghton Mifflin Co)
[13] deAguiar F S, Bernardes L b and Goulart Rosa S 1991 *J. Stat. Phys.* **64** 673
[14] Peitgen H-O, Jurgens H and Saupe D 1992 *Chaos and Fractals: New Frontiers of Science* (New York: Springer)
[15] Wu X N and Wu F Y 1989 *J. Phys. A: Math. Gen.* **22** L1031
[16] Monroe J L 1992 *J. Stat. Phys.* **67** 1185
[17] Ruelle D 1983 *Physica* D **7** 40